# MORE ON THE LINEAR SEARCH PROBLEM

BY

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ABSTRACT

The linear search problem concerns a search made in the real line for a point selected according to a given probability distribution. The search begins at zero and is made by continuous motion with constant speed along the line, first in one direction and then the other. The problem is to search in such a manner that the expected time required for finding the point according to the chosen plan of search is a minimum. This plan of search is usually conceived of as having a first step, a second, *etc.*, and in that case, this author has previously shown a necessary and sufficient condition on the probability distribution for the existence of a search plan which minimizes the expected searching time. In this paper, we define a notion of search in which there is no first step, but the steps are instead numbered from negative to positive infinity. These new rules change the problem, and under them, there is always a minimizing search procedure. In those cases which satisfy the earlier criterion, the solutions obtained are essentially the same as those obtained previously.

**Introduction.** In a recent paper by this author, [1], the linear search problem is discussed, and a necessary and sufficient condition is derived for the existence of a search procedure having minimal expected path. It is shown that when the left and right upper derivatives of the normalized distribution function are both infinite at 0, then no matter how small the first steps might be, it is nonetheless advantageous to add a yet smaller step before them, thus decreasing the expected path length. In this paper, we consider a modification of the definition of search procedure in which there is no first step. The procedures are conceived of as *beginning with an infinitesimal oscillation*, as defined below. Under this definition, which is a generalization of the concept of search procedure as defined in [1], a minimizing procedure exists for every distribution with finite first moment. Furthermore, if minimizing exists in the sense of [1], then the minimizing procedures derived here are the same ones, in a certain natural sense.

**Definitions and fundamental notions.** We begin with a probability distribution $F$ on the real line which has finite first moment $M_1 = M_1(F) = \int_{-\infty}^{+\infty} |t| \, dF(t)$. $F$ is assumed to be normalized to be continuous from the left in the left half-line, continuous from the right in the right half-line, and continuous at 0, for reasons discussed in [1]. In that paper, we define a search procedure as a sequence $x = \{x_i\}_{i=1}^{\infty}$ with

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$$\cdots \leqq x_4 \leqq x_2 \leqq 0 \leqq x_1 \leqq x_3 \leqq \cdots$$

or

$$\cdots \leqq x_3 \leqq x_1 \leqq 0 \leqq x_2 \leqq x_4 \leqq \cdots.$$

A function $X(x,t)$ was defined for $-\infty < t < +\infty$ as the length of the path from 0 to $t$ along the broken line running from 0 to $x_1$ to $x_2$ to $x_3$ *etc.* The expectation of this function, $X(x) = \int_{-\infty}^{+\infty} X(x,t)\,dF(t)$, was called the *expected path length* for the procedure $x$, and $m_0$ was the infimum of $X(x)$ for all such $x$. In this paper, we shall designate the set of such search procedures as $\mathfrak{X}_0$, and the functions $X(x,t)$ and $X(x)$ by $X_0(x,t)$ and $X_0(x)$ respectively. Then we have

$$m_0 = m_0(F) = \inf\{X_0(x) \mid x \in \mathfrak{X}_0\}.$$

We next define a *generalized search procedure.*

Let $x = \{x_i\}_{i=-\infty}^{+\infty}$, where

$$\cdots \leqq x_2 \leqq x_0 \leqq x_{-2} \leqq \cdots \leqq 0 \leqq \cdots \leqq x_{-1} \leqq x_1 \leqq x_3 \leqq \cdots,$$

and $\sum_{i=-\infty}^{0} |x_i| < \infty$. Let us imagine a point $t$ lying between $x_1$ and $x_3$. We can imagine a broken line from $t$ to 0 having for its vertices, in order, $t, x_2, x_1, x_0, x_{-1}, \cdots$. We can easily imagine the broken line traversed in the direction from $t$ to 0 (its length is $X_1(x,t) = |t| + \sum_{i=-\infty}^{2} 2|x_i|$), but it is harder to imagine motion in the opposite direction, since we stumble on the question: "What does one do *first*?" We say that the search plan $x$ so defined *begins with an infinitesimal oscillation.* As before, we define $X_1(x) = \int_{-\infty}^{+\infty} X_1(x,t)\,dF(t)$, where $X_1(x,t)$ is the distance from 0 to $t$ along the broken line whose vertices, in order, are

$$\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots.$$

As in [1], we define $x^-$ and $x^+$ so that $F(t) = 0$ if $t \leqq x^-$, $F(t) = 1$ if $t \geqq x^+$, while $0 < F(t) < 1$ for $x^- < t < x^+$. If, for instance, $-\infty < x^- < 0$, $x^+ = +\infty$, then we allow the possibility of a search procedure with $x_{j-1} = x^-$, $x_j = +\infty$, $\exists j$, and no entries $x_i$ for $i > j$. Similarly if $x^- = -\infty$, $0 < x^+ < +\infty$. In any case we do not allow any search procedures with entries $x_i$ for which $x^- \leqq x_i \leqq x^+$ does not hold.

For each $x \in \mathfrak{X}_0$, we can define a corresponding element $\bar{x} \in \mathfrak{X}_1$, by the following: If $\cdots \leqq x_4 \leqq x_2 \leqq 0 \leqq x_1 \leqq x_3 \cdots$, then

$$\bar{x}_i = \begin{cases} x_i & \text{if} \quad i > 0 \\ 0 & \text{if} \quad i \leqq 0 \end{cases}.$$

If $\cdots \leqq x_3 \leqq x_1 \leqq 0 \leqq x_2 \leqq x_4 \leqq \cdots$, then

$$\bar{x}_i = \begin{cases} x_{i+1} & \text{if} \quad i \geqq 0 \\ 0 & \text{if} \quad i < 0 \end{cases}.$$

Thus $\bar{x} \in \mathfrak{X}_2 = \{x \in \mathfrak{X}_1 \mid x_i = 0, \forall i < 0\}$, and

$$\forall y \in \mathfrak{X}_2, \exists x \in \mathfrak{X}_0 : \bar{x} = y.$$

For each $x \in \mathfrak{X}_0$, we have, of course, $X_0(x) = X_1(\bar{x})$. We define

$$m_1 = m_1(F) = \inf\{X_1(x) \mid x \in \mathfrak{X}_1\}.$$

For each $x \in \mathfrak{X}_1$, we have

$$\begin{aligned} X_1(x) = E(X_1(x,t)) &= \sum_{j=-\infty}^{+\infty} (-1)^{j+1} \int_{x_{j-2}}^{x_j} [|t| + \sum_{i=-\infty}^{j-1} 2|x_i|] \, dF(t) \\ &= \int_{-\infty}^{+\infty} |t| dF(t) + 2\sum_{i=-\infty}^{+\infty} |x_i|(1 - |F(x_i) - F(x_{i-1})|) \\ &= M_1(F) + 2\sum_{i=-\infty}^{\infty} x_i(F(x_i) - F(x_{i-1}) - (-1)^i) \end{aligned}$$

1. Lemma. Assume $x^- = -\infty$, $x^+ = +\infty$. Then for every $\varepsilon > 0$, $\exists B(\varepsilon) < \infty$ such that for all $x \in \mathfrak{X}_1$ with $X_1(x) < 2m_1$, we have

$$|x_j| < \varepsilon \;\Rightarrow\; |x_{j+1}| < B(\varepsilon).$$

**Proof.** Let $P_\varepsilon = \min(\Pr(t > \varepsilon), \Pr(t < -\varepsilon))$. Assume $x_j \geqq 0$; the other case is dual. Then $X_1(x,t) > 2|x_{j+1}|, \forall t > x_j$. Thus

$$2|x_{j+1}| \cdot P_\varepsilon < \int_\varepsilon^{+\infty} X_1(x,t) \, dF(t) < \int_{-\infty}^{+\infty} X_1(x,t) \; dF(t) < 2m_1,$$

so that $|x_{j+1}| < (m_1/P_\varepsilon) = B(\varepsilon)$. Q.E.D.

The same proof will give

2. Lemma. If $x^- < a \leqq b < x^+$, then $\exists\, B(a,b) < \infty$ such that for every $x \in \mathfrak{X}_1$ with $X_1(x) < 2m_1$,

$$x_j \in [a,b] \;\Rightarrow\; |x_{j+1}| < B(a,b).$$

3. Corollary (of Lemma 1). If $x^- = -\infty$, $x^+ = +\infty$, then for every $\varepsilon > 0$, $\exists\, C(\varepsilon) < \infty$ such that for every $x \in \mathfrak{X}_1$ with $X_1(x) < 2m_1$,

$$0 \leqq |x_j| < \varepsilon \;\Rightarrow\; |x_{j+2}| < C(\varepsilon).$$

**Proof.** $|x_{j+1}| < B(\varepsilon)$, by Lemma 1. Thus

$$|x_{j+2}| < B(B(\varepsilon)) = C(\varepsilon).$$ Q.E.D.

4. Definition. Choose any number $0 < \varepsilon_0 < x^+$, to be fixed for the remainder of the paper. (Note that if $x^- \geqq 0$ or $x^+ \leqq 0$, the problem is completely trivial, as in [1]).

Let $n_0$ be chosen so that $x_{n_0+1} > \varepsilon_0$, $x_i \leqq \varepsilon_0$, $\forall i \leqq n_0$. For each $x \in \mathfrak{X}_1$, define a new search plan $\tilde{x}$ by $\tilde{x}_i = x_{i-n_0}$. Then $X_1(x) = X_1(\tilde{x})$. $\tilde{x}$ is called the *normalized form of* $x$.

5. Lemma. Let $0 < \varepsilon < \varepsilon_0$. Then there is an $n = n(\varepsilon, F) > 0$ such that for every $x \in \mathfrak{X}_1$ with $X_1(x) < 2m_1$, $|\tilde{x}_i| < \varepsilon, \forall i < -n$.

**Proof.** Let $P = \Pr(t > \varepsilon_0)$, and let $k > 0$ be chosen so that either $|\tilde{x}_{-2k}| > \varepsilon$ or $|\tilde{x}_{-2k+1}| > \varepsilon$.

Then $\varepsilon < |\tilde{x}_{-2k}| + |\tilde{x}_{-2k+1}| \leqq |\tilde{x}_{-2k+2}| + |\tilde{x}_{-2k+3}| \leqq \cdots \leqq |\tilde{x}_{-2}| + |\tilde{x}_{-1}|$.

Thus $k\varepsilon < \sum_{i=-2k}^{-1} |\tilde{x}_i|$. For every $t > \varepsilon_0$, $t > \tilde{x}_{-1}$, so that

$$X_1(\tilde{x}, t) \;>\; 2\sum_{i=-\infty}^{0} |\tilde{x}_i| \;>\; 2\sum_{i=-2k}^{-1} |\tilde{x}_i| \;>\; 2k\varepsilon.$$

Thus we have

$$2k\varepsilon \cdot P < \int_{\varepsilon_0}^{+\infty} X_1(\tilde{x}, t)\, dF(t) < \int_{-\infty}^{+\infty} X_1(\tilde{x}, t)\, dF(t)$$
$$= X_1(\tilde{x}) = X_1(x) < 2m_1.$$

It follows that $2k < (2m_1/\varepsilon P)$, and therefore $2k - 1$ is also. Q.E.D.

6. Lemma. *If $x^- < a \leqq 0 \leqq b < x^+$, then $\exists n = n(a, b, F) > 0$ such that for all $x \in \mathfrak{X}_1$ with $X_1(x) < 2m_1$,*

$$\tilde{x}_j \notin [a, b], \forall j > n.$$

**Proof.** Let $P = \min(\Pr(t < a), \Pr(t > b))$. Choose $j > 0$ and assume that $\tilde{x}_{2j} \varepsilon [a, b]$. Then

$$\varepsilon_0 < |\tilde{x}_1| + |\tilde{x}_2| \leqq |\tilde{x}_3| + |\tilde{x}_4| \leqq \cdots \leqq |\tilde{x}_{2j-1}| + |\tilde{x}_{2j}|.$$

Thus, for $t < \tilde{x}_{2j}$, $X_1(\tilde{x}, t) \geqq j\varepsilon_0$ and

$$j\varepsilon_0 \cdot P < \int_{-\infty}^{a} X_1(\tilde{x}, t) dF(t) < \int_{-\infty}^{+\infty} X_1(\tilde{x}, t) dF(t) = X_1(x) < 2m_1,$$

yielding $j < (2m_1/\varepsilon_0 P)$.

If $\tilde{x}_{2j-1} \varepsilon [a, b]$, the same analysis shows that $j\varepsilon_0 P < \int_b^{+\infty} X_1(\tilde{x}, t)\, dF(t) < 2m_1$. Thus, in either case, $j < (2m_1/\varepsilon_0 P)$, and $\tilde{x}_k \notin [a, b]$ if $k > (4m_1/\varepsilon_0 P)$. Q.E.D.

7. Theorem. *If $x^- = -\infty$, $x^+ = +\infty$, then $\exists y \in \mathfrak{X}_1 : X_1(y) = m_1$.*

**Proof.** Let a sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of $\mathfrak{X}_1$ be chosen so that $X_1(x^{(n)}) \to m_1$ as $n \to \infty$. We note that $X_1(\tilde{x}^{(n)}) \to m_1$ also, so that it will not disturb the generality of the proof if we assume $x^{(n)} = \tilde{x}^{(n)}, \forall n$. Let each $x^{(n)}$ be designated as

$$\{x_i^{(n)}\}_{i=-\infty}^{+\infty}.$$

Then $0 \leqq \cdots \leqq x_{-3}^{(n)} \leqq x_{-1}^{(n)} \leqq \varepsilon_0, \forall n$, $0 \leqq \cdots \leqq |x_{-2}^{(n)}| \leqq |x_0^{(n)}| \leqq B(\varepsilon_0), \forall n$, and $|x_1^{(n)}| < B(B(\varepsilon_0))$, $|x_2^{(n)}| < B(B(B(\varepsilon_0)))$, etc. $\forall n$, so that for each $i$, $\{x_i^{(n)}\}_{n=1}^{\infty}$ is a bounded sequence, and thus contains a convergent subsequence. By the diagonal process, we extract a subsequence $\{x^{(n_j)}\}_{j=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ such that $\{x_i^{(n_j)}\}_{j=1}^{\infty}$ converges for each $i$, and such that $X_1(x^{(n_j)}) < 2m_1, \forall j$. We can assume without loss of generality that the sequence $\{x^{(n)}\}_{n=1}^{\infty}$ is actually the chosen subsequence. For each $i$, let $y_i = \lim_{n\to\infty} x_i^{(n)}$. Then $\cdots \leqq y_2 \leqq y_0 \leqq y_2 \cdots \leqq 0 \leqq \cdots \leqq y_{-1} \leqq y_1 \leqq \cdots$. Furthermore, for each $-\infty < a \leqq 0 \leqq b < +\infty$, we have $y_i \notin (a,b)$ for $i > n(a,b,F)$, so that $|y_i| \to \infty$ as $i \to +\infty$. Also, $|y_{-i}| \leqq \varepsilon$ if $i > n(\varepsilon, F)$, so that $y_i \to 0$ as $i \to -\infty$. Finally, if we set $P = \Pr(t > \varepsilon_0)$, we have

$$X_1(x^{(n)}, t) \geqq \sum_{i=-\infty}^{0} 2|x_i^{(n)}|, \forall t > \varepsilon_0, \forall n.$$

Thus

$$P \cdot \sum_{i=-\infty}^{0} 2|x_i^{(n)}| < \int_0^{+\infty} X_1(x^{(n)}, t)\, dF(t) < X_1(x^{(n)}) < 2m_1,$$

and

$$\sum_{i=-\infty}^{0} |x_i^{(n)}| < \frac{m_1}{P}, \forall n.$$

Therefore $\sum_{i=-\infty}^{0} |y_1| \leqq (m_1/P) < \infty$, and $y \in \mathfrak{X}_1$. To show that $X_1(y) = m_1$, choose a $\delta > 0$, and any $k$ large enough so that $\sum_{i=-\infty}^{-2k} |y_i| < \delta$. For each $n$, we now define a $w^{(n)} \in \mathfrak{X}_1$ as follows:

$$w_i^{(n)} = \begin{cases} x_i^{(n)} & \text{if} \quad i < -2k \\ (-1)^{i+1} \max(|x_i^{(n)}|, |y_i|) & \text{if} \quad -2k \leqq i \leqq 2k+1 \\ (-1)^{i+1} \max(|x_i^{(n)}|, |w_{i-2}^{(n)}|) & \text{if} \quad i > 2k+1. \end{cases}$$

Then $w_i^{(n)} = x_i^{(n)}$ for all but at most $s_k$ values of $i$, where $s_k = 2k+1+n(y_{2k}, y_{2k+1}, F)$. Choose any $\varepsilon > 0$ with $s_k \varepsilon < \delta$. Since $x_i^{(n)} \to y_i, \forall i$, we know that for all $n$ large enough, say all $n > n_1$, we have $|w_i^{(n)} - x_i^{(n)}| < \varepsilon$, $\forall i$. Then

$$X_1(w^{(n)}, t) < X_1(x^{(n)}, t) + s_k \varepsilon, \forall -\infty < t < +\infty, \forall n > n_1.$$

It follows that for $n > n_1$, $X_1(w^{(n)}) < X_1(x^{(n)}) + s_k\varepsilon$. Note that $w_i^{(n)} \to y_i$ as $n \to \infty, \forall i$, and define $v^{(n)} = \{v_i^{(n)}\} \in \mathfrak{X}_1$ by

$$v_i^{(n)} = \begin{cases} y_i & \text{if } i < -2k \\ w_i^{(n)} & \text{if } i \geqq -2k. \end{cases}$$

Then $X_1(v^{(n)}) < X_1(w^{(n)}) + 2\delta, \forall n$, since

$$\sum_{i=-\infty}^{-2k} 2|y_i| - \sum_{i=-\infty}^{-2k} 2|w_i^{(n)}| \leqq \sum_{i=-\infty}^{-2k} 2|y_i| < 2\delta.$$

Thus we have $X_1(v^{(n)}) < X_1(x^{(n)}) + s_k\varepsilon + 2\delta, \forall n$. We see immediately that

$$X_1(v^{(n)}, t) \to X_1(y, t)$$

uniformly for $y_{2k} \leqq t \leqq y_{2k+1}$, and thus

$$\begin{aligned}\int_{y_{2k}}^{y_{2k+1}} X_1(y,t)\ dF(t) &= \lim_{n\to\infty} \int_{y_{2k}}^{y_{2k+1}} X_1(v^{(n)}, t)\, dF(t) \\ &\leqq \limsup_{n\to\infty} X_1(v^{(n)}) \\ &\leqq \limsup_{n\to\infty} X_1(x^{(n)}) + s_k\varepsilon + 2\delta \\ &< m_1 + 3\delta.\end{aligned}$$

Since this inequality holds for all $k$ large enough, and $|y_i| \to \infty$ as $i \to +\infty$, we have

$$X_1(y) = \int_{-\infty}^{\infty} X_1(y,t)\, dF(t) \leqq m_1 + 3\delta.$$

Since $\delta$ is arbitrary, we have $X_1(y) \leqq m_1$, which gives us $X_1(y) = m_1$, by definition of $m_1$. Q.E.D.

8. Theorem. *If* $-\infty < x^- < 0$, $x^+ = +\infty$, *then* $\exists y \in \mathfrak{X}_1 : X_1(y) = m_1$.

**Proof.** Let $\{x^{(n)}\}_{n=1}^{\infty}$ be chosen so that $X_1(x^{(n)}) \to m_1$ as $n \to \infty$. Again, we can assume $x^{(n)} = \tilde{x}^{(n)}, \forall n$. Let $x^{(n)} = \{x_i^{(n)}\}_{i=-\infty}^{k_n}, \forall n$, where $-\infty < k_n \leqq +\infty$. For each $i$, consider the sequence $\{x_i^{(n)}\}_{n=1}^{\infty}$, where $x_i^{(n)} = +\infty$ if $i > k_n$. Either $\limsup_{n\to\infty} x_i^{(n)} < +\infty, \forall i$, or else $\exists k : \limsup_{n\to\infty} x_k^{(n)} = +\infty$, $\limsup_{n\to\infty} x_i^{(n)} < +\infty$, $\forall i < k$.

In the first case, we define $y$ as in Theorem 7, by a diagonal procedure on the $x^{(n)}$. It is clear from Lemma 6 that $y_{2k} \to x^-$, $y_{2k+1} \to +\infty$, as $k \to +\infty$. Then the proof of Theorem 7 employed verbatim will show that for arbitrary $\delta > 0$ and all $k$ large enough,

$$\int_{y_{2k}}^{y_{2k-1}} X_1(y,t)\, dF(t) \leqq m_1 + 3\delta.$$

Thus, as before, we have

$$X_1(y) = \int_{-\infty}^{+\infty} X_1(y,t) dF(t) = \int_{x^-}^{+\infty} X_1(y,t) dF(t) \leqq m_1,$$

and thus $X_1(y) = m_1$.

On the other hand, if the opposite case holds, then a subsequence $\{x^{(n_j)}\}_{j=1}^{\infty}$ cn be chosen from $\{x^{(n)}\}_{n=1}^{\infty}$ so that $x_k^{(n_j)} \to +\infty$ as $j \to \infty$, each sequence $\{x^{(n_j)}\}$ converges as $j \to \infty$ for $i < k$, and $X_1(x^{(n_j)}) < 2m_1, \forall j$. We can assume that the sequence $\{x^{(n)}\}_{n=1}^{\infty}$ itself satisfies all these conditions. Referring back to the proof of Theorem 8 in [1], we see that $\Pr(t < x_{k-1}^{(n)}) \to 0$ as $n \to \infty$, so that *a fortiori*, $y_{k-1} = x^-$. We now show, as in [1], that

$$\int_{x_{k-1}^{(n)}}^{x_k^{(n)}} X_1(y,t)\,dF(t) \leqq m_1,$$

so that $X_1(y) = \int_{x^-}^{+\infty} X_1(y,t)\,dF(t) \leqq m_1$, and $X_1(y) = m_1$ in this case also. Q.E.D.

9. COROLLARY. *If* $x^- = -\infty$, $0 < x^+ < +\infty$, *then* $\exists\, y \in \mathfrak{X}_1 \colon X_1(y) = m_1$.

**Proof.** Clear by symmetry.

10. THEOREM. *If* $-\infty < x^- < 0 < x^+ < +\infty$, *then* $\exists\, y \in \mathfrak{X}_1 \colon X_1(y) = m_1$.

**Proof.** As before, we choose a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ out of $\mathfrak{X}_1$ with $X_1(x^{(n)}) \to m_1$ as $n \to \infty$. We again assume that $x^{(n)} = \tilde{x}^{(n)}$, $\forall n$. We note that either

$$x^- < \liminf_{n\to\infty} x_i^{(n)}, \limsup_{n\to\infty} x_i^{(n)} < x^+, \ \forall\, i,$$

or else there is at least a value of $i$, call it $k$, such that one of the inequalities fails.

In the first case, the analysis of Theorems 7 and 8 will again show that a sequence $y$ extracted in the indicated way will have $X_1(y) = m_1$.

In the contrary case, assume that $\{x_k^{(n)}\}$ has a subsequence converging to $x^-$; the case of convergence to $x^+$ is dual. Then a subsequence $\{x^{(n_j)}\}_{j=1}^{\infty}$ can again be chosen so that $x_k^{(n_j)} \to x^-$ as $j \to \infty$, $\{x_i^{(n_j)}\}$ converges for each $i < k$, and $X_1(x^{(n_j)}) < 2m_1, \forall j$. Again, we assume $\{x^{(n_j)}\}_{j=1}^{\infty} = \{x^{(n)}\}_{n=1}^{\infty}$. Define $y_i$ as before for $i < k$, with $y_k = x^-$ and $y_{k+1} = x^+$. The previous analysis will now show that $X_1(y) = m_1$. Q.E.D.

We have now shown that in all cases, $\exists\, y \in \mathfrak{X}_1 : X_1(y) = m_1$. In [1], we showed that under certain circumstances, we have $\exists\, y \in \mathfrak{X}_0 : X_0(y) = m_0$. What is the relationship between these results?

11. LEMMA. $m_0(F) = m_1(F)$

**Proof.** For every $x \in \mathfrak{X}_0$, $x \in \mathfrak{X}_2$ and $X_0(x) = X_1(x)$. Thus

$$\begin{aligned} m_0 = \inf\{X_0(x) \mid x \in \mathfrak{X}_0\} &= \inf(X_1(x) \mid x \in \mathfrak{X}_2) \\ &\geqq \inf(X_1(x) \mid x \in \mathfrak{X}_1) \\ &= m_1 \quad . \end{aligned}$$

On the other hand, let $x \in \mathfrak{X}_1$ and choose $\delta > 0$ arbitrarily. We have $x_i \to 0$ as $i \to -\infty$. Thus, we can choose a $k$ so that

$$-\delta < x_{k+2} \leqq 0 \leqq x_{k+1} < \delta.$$

Let $y \in \mathfrak{X}_0$ be defined by

$$y_i = x_{k+i}, \ \forall\, i \geqq 1.$$

Then for each $-\infty < t < +\infty$, we have

$$X_0(y,t) < X_1(x,t) + 4\delta.$$

Since $m_0 \leqq X_0(y,t)$, we have

$$m_0 - 4\delta < X_1(x,t), \forall x \in \mathfrak{X}_1.$$

Thus $m_0 - 4\delta \leqq m_1$. Since $\delta$ is arbitrary, $m_0 \leqq m_1$, giving $m_0 = m_1$. Q.E.D.

In [1], we define

$$F^-(0) = \lim_{t \to 0-} \frac{F(t) - F(0)}{t}, \quad F^+(0) = \lim_{t \to 0+} \frac{F(t) - F(0)}{t}.$$

If at least one of these is finite, then $\exists\, y \in \mathfrak{X}_0$ with $X_0(y) = m_0$. There is no need for $y$ to be unique, of course. Under Theorems 7, 8 and 9 of this paper, we can find $y \in \mathfrak{X}_1$ with $X_1(y) = m_1 = m_0$. Are all the $y \in \mathfrak{X}_1$ with this property essentially representatives of elements of $\mathfrak{X}_0$? The answer is "yes", as seen from the next theorem.

12. Theorem. *Assume that* $F^+(0) < \infty$. *Let* $y \in \mathfrak{X}_1 : X_1(y) = m_0$. *Then* $\exists - \infty < k < +\infty$: $y_i = 0, \forall i \leqq k$.

**Proof.** Assume not. Then it is easily seen that $y_i \neq 0, \forall -\infty < i < +\infty$. Choose $D > 0$ with $F^+(0) < D < \infty$, and let $K > 0$ be chosen satisfying

1° $\dfrac{F(t) - F(0)}{t} < D, \forall 0 < t < K$

2° $F(K) - F(-K) < \dfrac{1}{2}$,

3° $K < \dfrac{1}{2D}$.

Since $\sum_{i=-\infty}^{0} |y_i| < \infty$, there must be an odd, negative number $k$ with

$$y_k - y_{k+1} = |y_k| + |y_{k+1}| < K.$$

We shall show that $y_k = y_{k-1} = 0$.

Define $x \in \mathfrak{X}_1$ by

$$x_i = \begin{cases} y_i, & \forall i > k, \\ y_{i-2}, & \forall i \leqq k. \end{cases}$$

Then

$$\begin{aligned} X_1(y)-X_1(x) = {} & 2[|y_{k-1}|(1-F(y_{k-2})+F(y_{k-1}))+|y_k|(1-F(y_k)+F(y_{k-1})) \\ & +|y_{k+1}|(1-F(y_k)+F(y_{k+1}))-|y_{k+1}|(1-F(y_{k-2})+F(y_{k+1}))] \\ = {} & 2[(|y_{k-1}|+|y_k|)(1-F(y_{k-2})+F(y_{k-1})) \\ & +|y_k|(-F(y_k)+F(y_{k-2})) \\ & +|y_{k+1}|(-F(y_k)+F(y_{k-2}))]. \\ = {} & 2[(y_k-y_{k-1})(1-F(y_{k-2})+F(y_{k-1})) \\ & -(y_k-y_{k+1})(F(y_k)-F(y_{k-2}))]. \end{aligned}$$

We observe the following:

(a) $y_k \leqq y_k - y_{k-1}$, with equality only if $y_{k-1} = 0$.

(b) $y_{k-2} \leqq y_k \leqq y_k - y_{k+1} < K$, and

$y_{k-1} \geqq y_{k+1} \geqq y_{k+1} - y_k > -K$, so that

$F(y_{k-2}) - F(y_{k-1}) \leqq F(K) - F(-K) < \frac{1}{2}$,

and $1 - F(y_{k-2}) + F(y_{k-1}) > \frac{1}{2}$.

(c) $F(y_k) - F(y_{k-2}) \geqq F(y_k) - F(0) \geqq Dy_k$,

with equality holding only if $y_k = 0$.

Thus,

$$X(y) - X(x) \geqq 2[y_k \cdot \frac{1}{2} - K \cdot Dy_k],$$

with equality holding only if $y_k = y_{k-1} = 0$. However, $X(y) - X(x) \leqq 0$ by assumption on $y$, while $KD < \frac{1}{2}$ by definition on $K$, so that

$$\frac{1}{2} y_k - KDy_k \geqq 0.$$

It follows that $y_k = y_{k-1} = 0$. Q.E.D.

One important feature of an absolute minimum, aside from aesthetic considerations, lies in the fact that a recursion formula for the entries can sometimes be obtained by partial differentiation. Let $x^{(0)} \in \mathfrak{X}_1$, with $X_1(x^{(0)}) = m_1$. Assume that $F$ is differentiable at each $x_i^{(0)}$, $-\infty < i < +\infty$. Then $X_1(x)$ is a differentiable function of each $x_i$ at $x = x^{(0)}$, and

$$\frac{\partial X_1(x^{(0)})}{\partial x_i} = 0, \ \forall -\infty < i < +\infty.$$

Since $(\partial(X_1(x))/\partial x_i) = 2[(x_{i+1} - x_i)F'(x_i) - (-1)^i - F(x_i) + F(x_{i-1})]$, we have

$$x_{i+1} = x_i + \frac{F(x_{i-1}) - F(x_i) - (-1)^i}{F'(x_i)}$$

and

$$F(x_{i-1}) = F(x_i) + (-1)^i + (x_i - x_{i+1})F'(x_i).$$

Thus, under proper differentiability conditions, we can derive all the $x_i$ of a minimal solution if we have two consecutive ones. Clearly, if $F$ is a strictly increasing function, the two equations above will give us all the $x_i$. To extend the same observation to distributions which are not strictly increasing, we note that of all the values of $x$ for which $F(x)$ takes a given value, only that value of $x$ having the smallest absolute value can appear as an entry in a minimal search procedure. The formulae hold as well for $x \in \mathfrak{X}_1$, as for $x \in \mathfrak{X}_0$.

Although the problem as originally posed has as yet no solution in a useful sense, even for approximations, the analysis here is too delicate to carry over approximations, and the recurrence relations, which depend strongly on $F'$ (a very sensitive quantity) do not withstand the approximation process.

## Bibliography

1. Anatole Beck, *On the linear search problem*, Israel Journal of Math. **2** (1964), 221–228.*

2. Wallace Franck, *On the optimal search problem*, Technical Report No. 44, University of New Mexico, Oct. 1963.

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* Please note the following erratum in [1]: Equation 2° on page 224 should read:

$$2^\circ \quad \frac{F(t) - F(0)}{t} < D, \ \forall -K < t < 0.$$